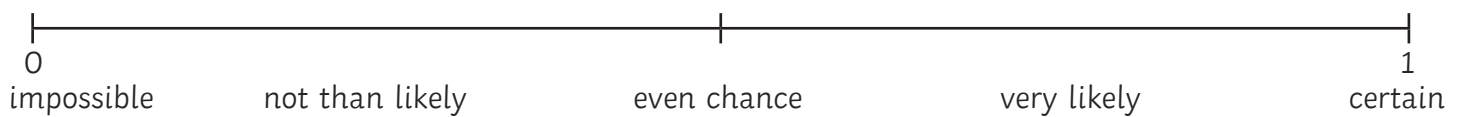


# Probability: Probability Scale

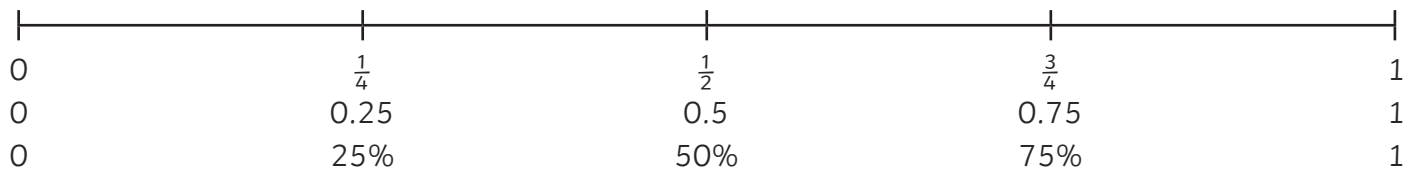
Probability is the mathematics of chance or the likelihood of something happening. We often use the language of chance in our day-to-day lives:

- definitely won't happen (impossible)
- definitely will happen (certain)
- not that likely
- very likely
- could go either way

We can show these possible outcomes on a probability scale, ranging from 0 (impossible) to 1 (certain).



If the range of probability lies between 0 and 1, how can we mathematically represent the different possible outcomes between impossible and certain?

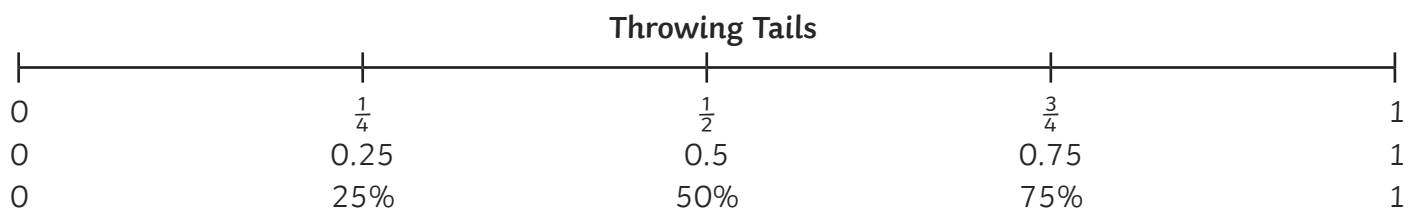


The probability scale (above) can be used to show the likelihood of any event, with probability given as either a fraction, decimal or percentage.

## Example

What is the probability of throwing tails on a fair coin? Show the probability on the scale below.

The probability of throwing tails is the same as the probability of throwing heads, so it would go in the exact middle of the scale:



To put this mathematically:  $P(\text{Tails}) = \frac{1}{2}$ , 0.5 or 50%

# Probability: Listing Outcomes

Listing outcomes is, quite simply, writing a list of all the possible outcomes of a probability experiment. The trick is to be methodical.

## Example 1

List all the possible outcomes of throwing two coins. To make it easier, let's use H for heads and T for tails.

The outcomes are: H,H H,T T,H T,T

Coin 2 \ Coin 1	Heads (H)	Tails (T)
Heads (H)	H,H	H,T
Tails (T)	T,H	T,T

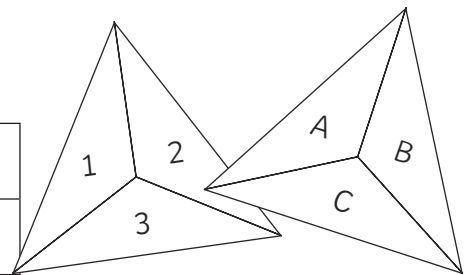


## Example 2

List all the possible outcomes of spinning two three-sided spinners.

The outcomes are: 1,A 1,B 1,C  
2,A 2,B 2,C  
3,A 3,B 3,C

Spinner 2 \ Spinner 1	1	2	3
A	1,A	2,A	3,A
B	1,B	2,B	3,B
C	1,C	2,C	3,C



From these lists it is now easy to work out the probabilities of different outcomes.

Remember: **P(outcome)** =  $\frac{\text{Number of times that outcome could happen}}{\text{Total number of all possible outcomes}}$

## Example 1 Find:

- a) P(both H) There is only one way for both coins to be heads and four possible outcomes in total. Therefore:  $P(\text{both H}) = \frac{1}{4}$
- b) P(H and T) There are two ways for the coins to be a head and a tail (H, T and T,H) and four possible outcomes in total. Therefore:  $P(\text{H and T}) = \frac{2}{4} = \frac{1}{2}$

## Example 2 Find:

- a) P(1 and A) There is only one way for the spinners to show 1 and A and nine possible outcomes in total. Therefore:  $P(1 \text{ and } A) = \frac{1}{9}$
- b) P(3) There are three ways for the first spinner to show 3 (3,A and 3,B and 3,C) and nine possible outcomes in total. Therefore:  $P(3) = \frac{3}{9} = \frac{1}{3}$

**Don't forget:** The sum of all outcome probabilities for each experiment will always be 1.

# Probability: Independent and Mutually Exclusive Events

Every separate event in a probability experiment has its own probability. Once we know all of these, we can work out the probability of more than one event happening. These might be independent events, where the outcome of one event has no effect on the other (the **and** rule), or mutually exclusive events, where the events cannot happen at the same (the **or** rule).

## Example

You have two three-sided spinners.

The **and** rule:

Let  $X$  = the probability of spinning 1

Let  $Y$  = the probability of spinning C

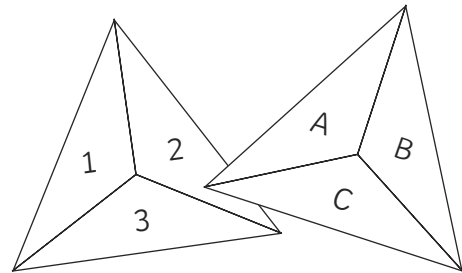
Find the probability that the two spinners will land on 1 and C together in one event.

i.e. Find  $P(X \text{ and } Y)$

$$P(X) = \frac{1}{3}$$

$$P(Y) = \frac{1}{3}$$

$$P(X \text{ and } Y) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$$



The **or** rule:

Find the probability that the spinner will land on A or B

i.e. Find  $P(A \text{ or } B)$

$$P(A) = \frac{1}{3}$$

$$P(B) = \frac{1}{3}$$

Therefore  $P(A \text{ or } B) = P(A) + P(B)$

$$P(A \text{ or } B) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

# Probability: **And/Or Rule** (Higher)

Be careful when using the and rule when events are dependent (conditional), and the first outcome affects the probability of the second.

## Example

We will select cards from a pack of 52 cards and not return them to the pack.

Let A = the probability of selecting a king

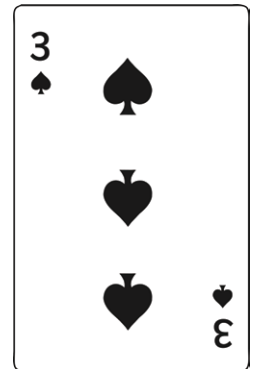
Let B = the probability of selecting a queen

Find the probability of selecting a king and then a queen.

i.e. Find  $P(A \text{ and } B)$

$$P(A) = \frac{4}{52} = \frac{1}{13} \quad P(B) = \frac{4}{51}^*$$

\* The second card is taken from a pack of 51 – one less now the first card has been taken.



**Look out for** using the or rule when events are not mutually exclusive (they can happen at the same time).

## Example

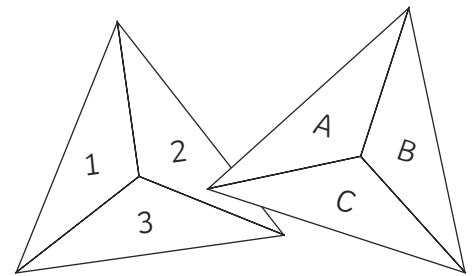
Let X = the probability of spinning a 1

Let Y = the probability of spinning a C

Find the probability of an outcome that includes a 1 or a C.

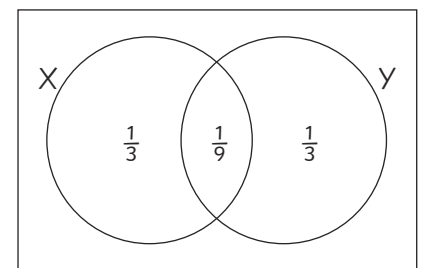
i.e. Find  $P(X \text{ or } Y)$

$$P(X) = \frac{1}{3} \quad P(Y) = \frac{1}{3}$$



**However**, there is also an outcome that contains **both** of these events,  $P(X \text{ and } Y)$ . This is when the spinners show 1,C. This outcome is counted twice in both of the probabilities for X and Y.

A Venn Diagram helps to show this:



To find  $P(X \text{ or } Y)$  we need to find  $P(X)$  **and**  $P(Y)$  **without**  $P(X \text{ and } Y)$ .

$$P(X \text{ or } Y) = P(X) + P(Y) - P(X \text{ and } Y)$$

$$P(X \text{ or } Y) = \frac{1}{3} + \frac{1}{3} - \frac{1}{9} = \frac{5}{9}$$

Finding the probability of only 1 **or** C (not both) as an outcome means getting rid of  $P(A \text{ and } B)$  completely from both  $P(A)$  and  $P(B)$ .

In this instance,  $P(A \text{ or } B) = P(A) - P(A \text{ and } B) + P(B) - P(A \text{ and } B)$ .

# Probability: Relative Frequency

Relative frequency is the number of times something happened in relation to the number of times it was tried. Relative frequency formula:

$$\text{Probability of specific outcome} = \frac{\text{Number of times outcome occurred}}{\text{Total number of attempts}}$$

## Example

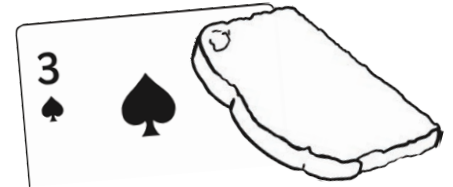
I decide to check the relative frequency of toast landing jam side up when I drop it. I drop it 25 times. It lands jam side up 16 times.

Therefore,  $P(\text{toast landing jam side up}) = \frac{16}{25}$  or 0.64  
(relative frequencies are usually written as decimals)

Using this relative frequency, I can make predictions:

If I drop my toast 500 times, I predict that the number of times it will land jam side up is  $500 \times 0.64 = 320$  times.

If I drop my toast 1000 times, I predict that the number of times it will land jam side up is  $1000 \times 0.64 = 640$  times.



## However...

How accurate do you think this is? How can we make relative frequency more accurate?

Remember: the more times you carry out an experiment, the more likely it is to tend towards a theoretical probability distribution. This is just a fancy way of saying the results get more accurate and we can feel more confident using them to predict theoretical situations - like we did with our 500 or 1000 toast drops.

Relative frequency is also used to check whether a sample is biased by making comparisons with the expected theoretical probability

## Example

My brother uses a special deck of cards for his card tricks.

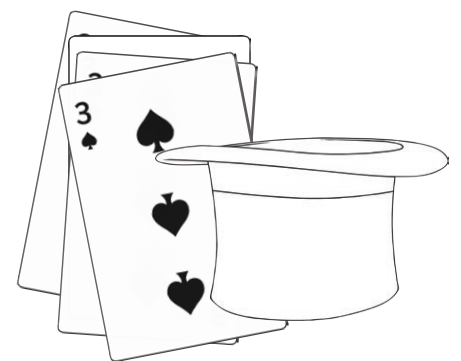
I wanted to see whether the deck was an unbiased, normal pack of cards, or whether there was something fishy going on.\*

I pulled out 50 cards (returning the card to the pack each time) and I pulled out 24 kings in total. Let's examine the evidence for selecting kings from decks of cards:

- Relative Frequency  $\frac{24}{50}$  or 0.48 (probability of almost one half!)
- Expected Theoretical Frequency  $\frac{1}{13}$  or 0.08 (to 2 dp)

In my brother's pack of cards, the probability of selecting a king is considerably higher than the expected probability of  $\frac{1}{13}$  which suggests the deck of cards is hugely biased.

It also means my brother is a rubbish magician.



\*OK, so I could have just looked through the cards to see, but where's the mathematical fun in that?

# Probability: Two-Way Tables

A two-way table compares the probabilities of different categories of data for the same thing. The two-way tables you are most likely to encounter are frequency tables and relative frequency tables.

## Frequency tables

Frequency is a fancy way of saying 'number'. As in, 'The frequency of mathematicians making up annoying words we have to remember instead of using the usual ones we already know, is quite high'.

## Example

Here is a two-way frequency table showing the favourite music of men and women.

	Rock	Pop
Men	24	17
Women	31	9

In a nutshell:

24 men like rock, 17 men like pop

31 women like rock, 9 women like pop

The trick to using these tables to calculate probabilities is to keep track of the totals for each column and row:

	Rock	Pop	<b>Total</b>
Men	24	17	<b>41 men</b>
Women	31	9	<b>40 women</b>
<b>Total</b>	<b>55 rock fans</b>	<b>26 pop fans</b>	<b>81 people</b>

Let's look at the different ways we can break this information down:

**Total number of:** Women (40), men (41), people (81), rock fans (55), pop fans (26)

**Rock preference by gender:** Women ( $\frac{31}{40}$ ), men ( $\frac{24}{41}$ )

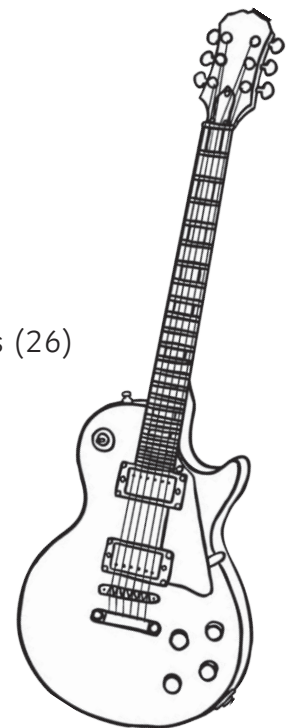
**Pop preference by gender:** Women ( $\frac{9}{40}$ ), men ( $\frac{17}{41}$ )

**Rock preference by total population:** Women ( $\frac{31}{81}$ ), men ( $\frac{24}{81}$ )

**Pop preference by total population:** Women ( $\frac{9}{81}$ ), men ( $\frac{17}{81}$ )

**Fraction of rock fans who are:** Women ( $\frac{31}{55}$ ), men ( $\frac{24}{55}$ )

**Fraction of pop fans who are:** Women ( $\frac{9}{26}$ ), men ( $\frac{17}{26}$ )



The questions you are given will help you identify which sets of figures to use to work out probability.

For example:

- What is the probability that a woman ( $_{40}$ ) selected will like pop? ( $\frac{9}{40}$ )
- What is the probability that a person ( $_{81}$ ) selected will like pop? ( $\frac{26}{81}$ )
- What is the probability that a person ( $_{81}$ ) selected will be a woman who likes pop? ( $\frac{9}{81}$ )

### Relative Frequency tables

These are the same as frequency tables except the data given is the probability of the event, not just the number (sorry, frequency). You use these probabilities to calculate different outcomes (using the and/or rules).

#### Example

Here is a two-way relative frequency table showing estimates for the probabilities of Ken and Barbie winning different lotteries (they need the dosh to renovate the beach house). They are independent events.

	Hospital Lotto	Community Centre Raffle
Ken	0.4	0.26
Barbie	0.09	0.5



Different ways of using this information include finding:

- The probability that Ken wins just the Lotto (0.4), just the Raffle (0.26), both the Lotto and the Raffle ( $0.4 \times 0.26$ ), or either the Lotto or the Raffle ( $0.4 + 0.26$ )
- The probability that Barbie wins just the Lotto (0.09), just the Raffle (0.5), both the Lotto and Raffle ( $0.09 \times 0.5$ ), or either the Lotto or the Raffle ( $0.09 + 0.5$ )
- The probability that Ken and Barbie both win the Lotto ( $0.4 \times 0.09$ )
- The probability that Ken or Barbie win the Raffle ( $0.26 + 0.5$ )

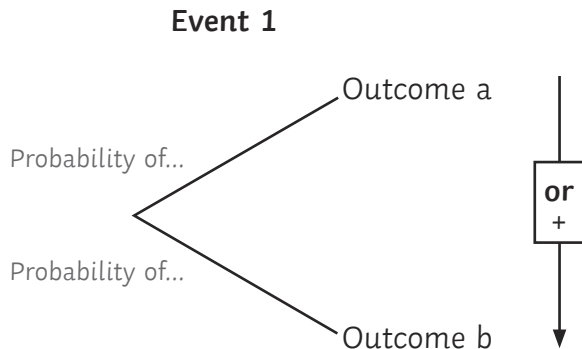
Creating two-way relative frequency tables from frequency is simple in theory: just use the relative frequency formula for the numbers you are given (frequency of outcome  $\div$  total number of possible outcomes).

However, it can get complicated if there is a choice of total numbers to use – if we look at the last example of the musical tastes of men and women, we can see that the relative frequencies can be calculated using the total number of men, total number of women, all people, all rock fans or all pop fans. If they are mean enough to give you this in an exam, read the question carefully and look for clues as to which fractions to use.

# Probability: Tree Diagram

These are great. I mean it. Use them wherever you can. Once you have the hang of them you can use them for everything and you will learn to love them.

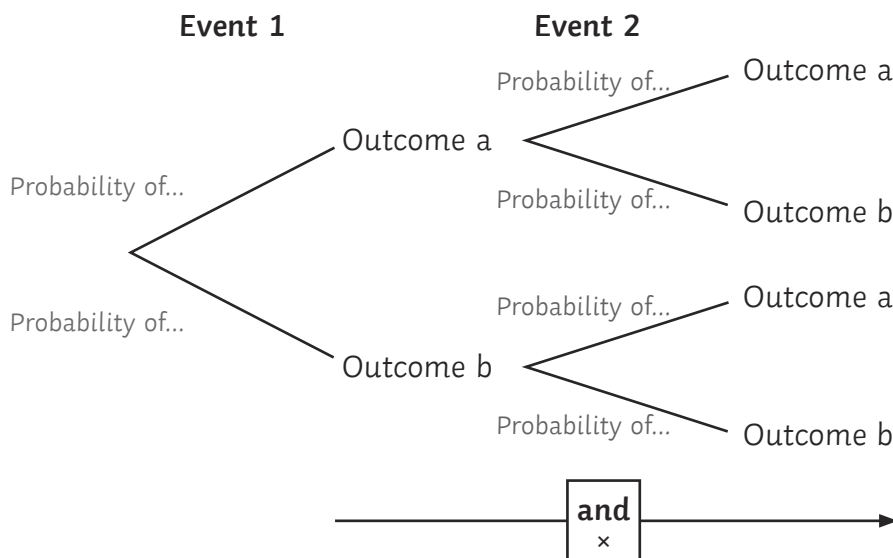
Let's break it down.



**Step 1:** list your possible outcomes (you might have more than two) and draw branches from the same point in a tree like fashion. Write the probability of each outcome by the 'branch'.

This diagram clearly shows  $P(\text{Outcome a})$  and the  $P(\text{Outcome b})$ . The  $P(\text{Outcome a or b})$  can be found by adding down the diagram.

What about a sequence of events where two outcomes occur? More tree action please!



**Step 2:** draw the possible outcome branches for the 2nd event from all the outcomes in the 1st event. Write the probability of each outcome by the 'branch'.

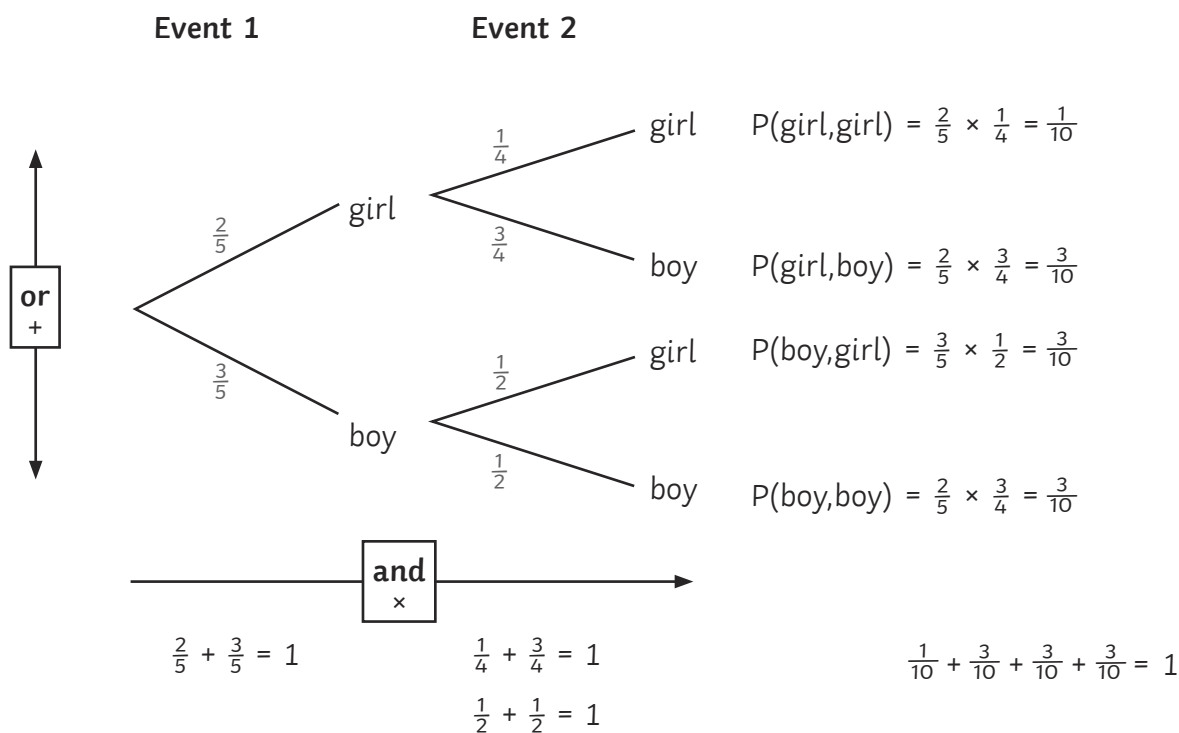
**Don't forget:** look out for conditional probabilities, when the second outcome probability is affected by the first (such as not replacing cards or beads when they have been selected).

Now this diagram clearly shows  $P(\text{Outcome a and a})$ ,  $P(\text{Outcome a and b})$ ,  $P(\text{Outcome b and a})$  and  $P(\text{Outcome b and b})$ . The  $P(\text{Outcome } \_ \text{ and } \_)$  can be found by multiplying across the branches.



**Example**

You are picking teams for dodgeball. There are 5 people left, 2 girls and 3 boys. One person is chosen at a time. Draw a tree diagram showing the different outcomes and probabilities for the next two selections.

**Things to notice:**

- Selecting a boy or a girl is a mutually exclusive event. The sum of the probabilities for each event will add up to 1.
- Once a person is selected in the 1st event, there are only four more people to choose from for the 2nd event, not 5, and hence the probability for event 2 changes (conditional).

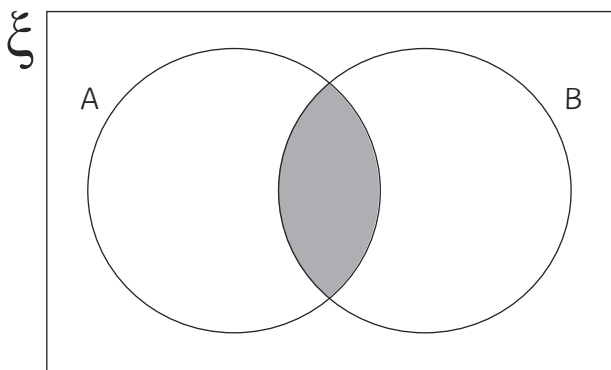
What is the probability that the two selections are the same sex -  $P(\text{girl, girl})$  or  $P(\text{boy, boy})$ ?

The **or** rule works down the tree, so simply add the probabilities you need together.

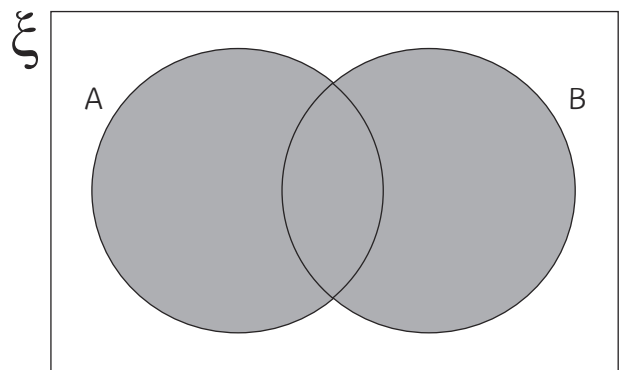
$$P(\text{girl, girl}) = \frac{1}{10}, P(\text{boy, boy}) = \frac{3}{10}$$

$$\text{So the } P(\text{selections are the same sex}) = \frac{1}{10} + \frac{3}{10} = \frac{2}{5}$$

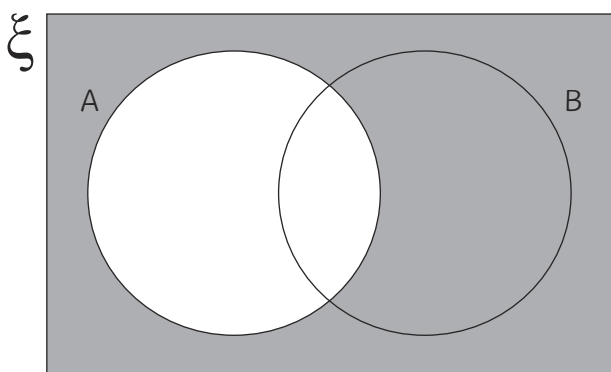
# Probability: Venn Diagram Notation



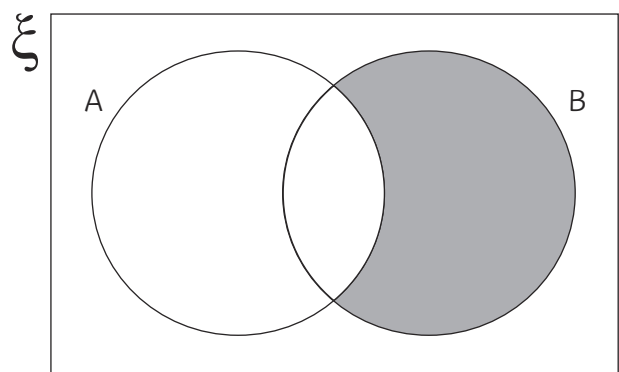
$A \cap B$  means the intersection of A and B



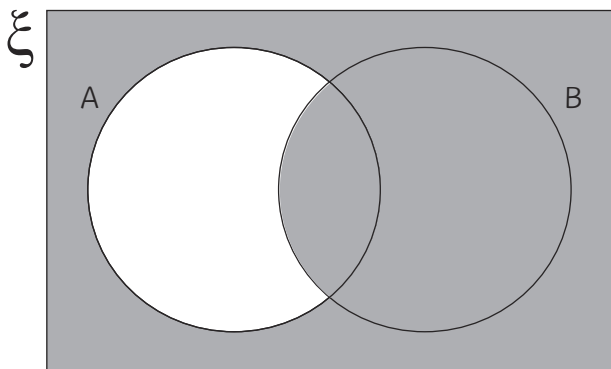
$A \cup B$  means the union of A and B



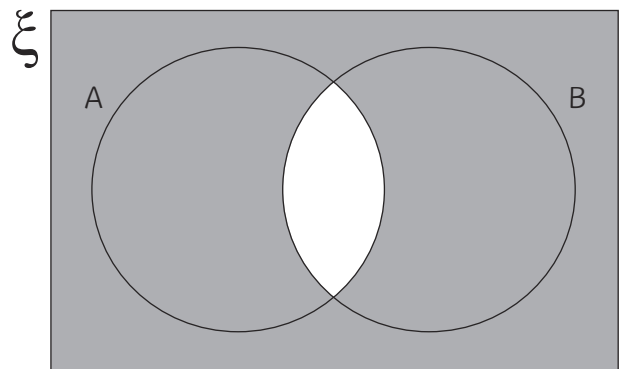
$A'$  means everything not in A



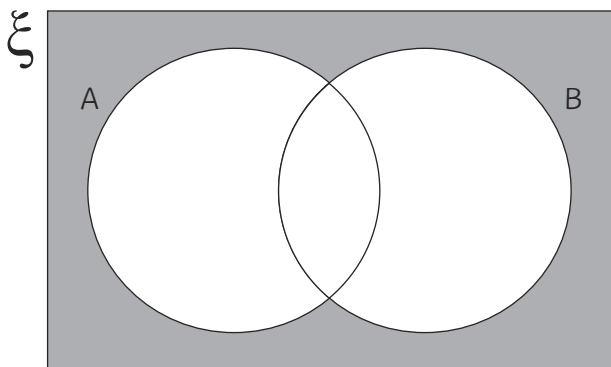
$A' \cap B$  means everything not in A that is in B



$A' \cup B$  means the union of  $A'$  and B



$(A \cap B)'$  means everything not in the intersection



$(A \cup B)' = A' \cap B'$  means everything not in the union